On a problem of W. Sierpinski

DARIUSZ FREJMAN and ALEKSANDER GRYTCZUK

Abstract. In this paper we dealt with the Diophantine equation $(x^2-1)^2 + (y^2-1)^2 = (z^2-1)^2$. We prove: if $x > 1, y > 1$ and $z = y+1$ then the only solution is $(x, y, z) = (10, 13, 14)$.

Let $t_n = \frac{n(n+1)}{2}$ be the n-th triangular number. K. Zarankiewicz (see [2], p. 53) has asked whether there exists a Pytagorean triangle whose sides are triangular numbers, i. e.

$$t_a^2 + t_b^2 = t_c^2. \tag{1}$$

The answer to this question is affirmative, because for $a = 132$, $b = 143$, $c = 164$ the triangular numbers $t_{132}, t_{143}, t_{164}$ satisfy (1). On the other hand we have $8t_n = (2n+1)^2 - 1$ and we see that (1) is equivalent to

$$\left((2a+1)^2-1\right)^2 + \left((2b+1)^2-1\right)^2 = \left((2c+1)^2-1\right)^2. \tag{2}$$

Thus the equation

$$(x^2-1)^2 + (y^2-1)^2 = (z^2-1)^2 \tag{3}$$

has a solution in odd natural numbers x, y, z, namely $x = 2a+1 = 265$, $y = 2b+1 = 287$, $z = 2c+1 = 329$. The equation (3) has also another solution in which not all numbers x, y, z are odd, namely $x = 10$. $y = 13$, $z = y+1 = 14$ (Cf. [2], p.54).

W. Sierpinski (see [2], p.54) writes that we do not know whether the equation (3) has infinitely many solutions in natural numbers greater than one.

In this connection we prove the following theorem:

Theorem. The Diophantine equation

$$(x^2-1)^2 + (y^2-1)^2 = (z^2-1)^2$$

has exactly one solution in natural numbers $x > 1$, $y > 1$, z, such that $z = y + 1$, namely

$$< x, y, z > = < 10, 13, 14 > .$$

In the proof of the Theorem we use the following:

Lemma. The Diophantine equation

$$3u^4 - 2v^2 = 1$$

has exactly two solutions in natural numbers u, v namely

$$< u, v > = < 1, 1 >, < 3, 11 > .$$

The proof of this Lemma has been given by R. T. Bumby in [1].

PROOF of the Theorem. Let $z = y + 1$ and suppose that the equaiton (3) has a solution in natural numbers x, y, z. Then we have

$$\left(x^2 - 1\right)^2 = (2y + 1)\left(2y^2 + 2y - 1\right) .$$

Let $d = \left(2y + 1, 2y^2 + 2y - 1\right)$ then we have

$$2y + 1 = dA, \quad 2y^2 + 2y - 1 = dB; \quad (A, B) = 1. \tag{5}$$

From (5) we have

$$d^2A^2 - 3 = 2d \cdot B. \tag{6}$$

By (6) it follows that $d \mid 3$, thus $d = 1$ or $d = 3$.

Let us consider the case $d = 1$. Then by (5) and (4) it follows that

$$\left(x^2 - 1\right)^2 = A \cdot B; \quad (A, B) = 1. \tag{7}$$

From (7) we obtain that there exist integers α, β such that $(\alpha, \beta) = 1$ and

$$A = \alpha^2, B = \beta^2, x^2 - 1 = \alpha \cdot \beta. \tag{8}$$

By (8) and (6) it follows that

$$\alpha^4 - 3 = 2\beta^2, \quad (\alpha, \beta) = 1. \tag{9}$$

From (9) we have that α is an odd number. If β is also an odd number then by (9) we have $2\beta^2 + 3 \equiv 5 \pmod 8$ and so $\alpha^2 \equiv 1 \pmod 8$ and we get a contradiction.

If β is an even number then we have $2\beta^2 + 3 \equiv 3 \pmod 4$ and $\alpha^2 \equiv 1 \pmod 4$ and similarly we obtain a contradiction mod 4.

Thus we obtain that the equation (9) has no solution in natural numbers α, β.

Now we can consider the case $d = 3$.

By (5) and (4) it follows that

$$\left(\frac{x^2-1}{3}\right)^2 = A \cdot B, \quad (A, B) = 1. \tag{10}$$

From (10) we obtain

$$A = \alpha^2, \quad B = \beta^2, \quad x^2 - 1 = 3\alpha\beta, \quad (\alpha, \beta) = 1. \tag{11}$$

By (11) and (6) it follows that

$$3\alpha^4 - 2\beta^2 = 1. \tag{12}$$

Applying Bumby's result (see Lemma) to (12) we have

$$< \alpha, \beta > = < 1, 1 >, < 3, 11 > . \tag{13}$$

If $\alpha = \beta = 1$ then $A = B = 1$ and $x^2 - 1 = 3$ thus $x = 2$. From (5) we have $2y + 1 = 3 \cdot 1$ thus $y = 1$, which contrardicts to our assumpiton $y > 1$.

Hence $\alpha = 3$, $\beta = 11$ and we obtain $A = \alpha^2 = 3^2 = 9$, $B = \beta^2 = 11^2 = 121$, $x^2 - 1 = d\alpha\beta = 3^2 \cdot 11 = 99$ thus $x = 10$. From (5) $2y + 1 = dA = 3 \cdot \alpha^2 = 3 \cdot 3^2 = 27$, thus $y = 13$. Because $z = y + 1$, thus $z = 14$. Hence $< x, y, z > = < 10, 13, 14 >$ and the proof of the Theorem is complete.

References

[1] R. T. BUMBY, The Diophantine equation $3x^4 - 2y^2 = 1$, *Math. Scand.*, **21** (1967), 144–148.

[2] W. SIERPINSKI, Elementary Theory of Numbers, *PWN Warszawa*, (1987).